Application of HCR's Cosine Formula

Minimum distance between any two points on a sphere

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We know that the length of minor great circle arc joining any two arbitrary points on a sphere of finite radius is the minimum distance between those points. Here we are interested in finding out the minimum distance or great circle distance between any two arbitrary points on a spherical surface of finite radius (like globe) for the given values of latitudes & longitudes.

Let there any two arbitrary points $\boldsymbol{A(\phi_1, \lambda_1)}$ & $\boldsymbol{B(\phi_2, \lambda_2)}$ on the surface of sphere of radius $\boldsymbol{R}$ & centre at the point O. The **angles of latitude** $\boldsymbol{\phi_1}$ & $\boldsymbol{\phi_2}$ are measured from the equator plane & the **angles of longitude** $\boldsymbol{\lambda_1}$ & $\boldsymbol{\lambda_2}$ are measured from a reference plane OPQ in the anticlockwise direction. Here, we are to find out the length of **great circle arc AB** joining the given points A & B. Draw the great circle arcs passing through the points A & B which intersect each other at the peak (pole) point P & intersect the equatorial line orthogonally at the points D & C respectively. (As shown by the dotted arcs PD & PC in the figure 1)

Join the points A, B, C & D by the dotted straight lines through the interior of sphere to get a **plane quadrilateral ABCD**.

Now, the angle between the great circle arcs AD & DC, subtending the angles $\alpha = \phi_1$ & $\beta = \lambda_2 - \lambda_1$ respectively at the centre O of the sphere, meeting each other at the common end point D, is $\theta = \pi/2$, hence the (plane) angle $\measuredangle ADC = \theta_D$ between the chords AD & DC of great circle arcs AD & DC is given by **HCR's cosine formula** as follows

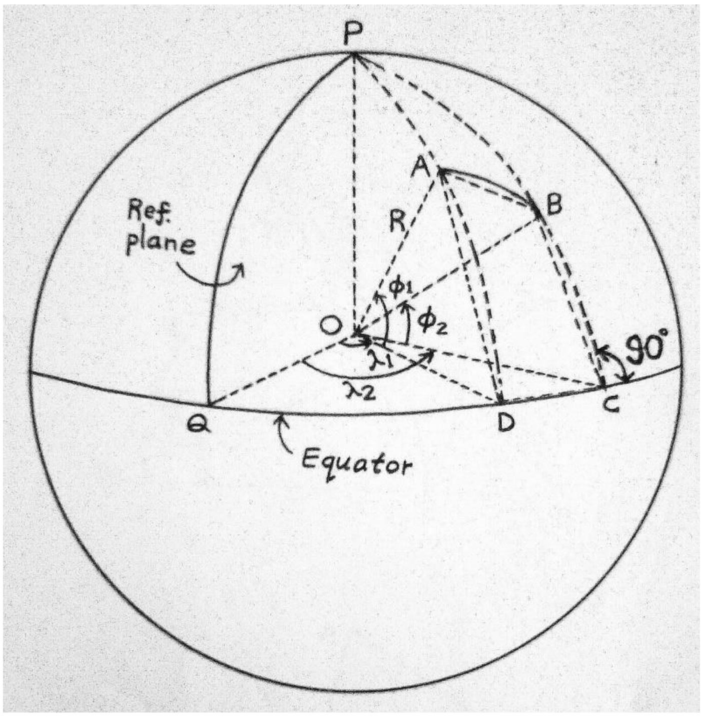

Figure 1: The dotted great circle arcs passing through two given points A & B, intersecting each other at the peak (pole) point P, meet the equatorial line orthogonally at the points D & C respectively on a spherical surface of finite radius $\boldsymbol{R}$

$$\boldsymbol{\cos\theta_p = \sin\frac{\alpha}{2}\sin\frac{\beta}{2} + \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\theta} \qquad \forall\ \ 0 < \alpha, \beta, \theta \le \pi \ \Rightarrow\ 0 \le \theta_p < \theta$$

Now, setting the corresponding values in the above formula, we get

$$\cos\theta_D = \sin\frac{\phi_1}{2}\sin\left(\frac{\lambda_2 - \lambda_1}{2}\right) + \cos\frac{\phi_1}{2}\cos\left(\frac{\lambda_2 - \lambda_1}{2}\right)\cos\frac{\pi}{2}$$

$$\boldsymbol{\cos\theta_D = \sin\frac{\phi_1}{2}\sin\left(\frac{\lambda_2 - \lambda_1}{2}\right)} \qquad \forall\ \ 0 \le \theta_D < \frac{\pi}{2} \qquad \ldots\ldots\ldots\ldots\ldots \ (1)$$

Similarly, setting the values of angles $\alpha = \phi_2$ & $\beta = \lambda_2 - \lambda_1$ & $\theta = \pi/2$ in the formula, the (plane) angle $\measuredangle BCD = \theta_C$ between the chords BC & CD of great circle arcs AD & DC, meeting each other orthogonally at the common end point C, is given as

$$cos\theta_C = sin\frac{\phi_2}{2}sin\left(\frac{\lambda_2-\lambda_1}{2}\right) + cos\frac{\phi_2}{2}cos\left(\frac{\lambda_2-\lambda_1}{2}\right)cos\frac{\pi}{2}$$

$$\boldsymbol{cos\theta_C = sin\frac{\phi_2}{2}sin\left(\frac{\lambda_2-\lambda_1}{2}\right)} \qquad \forall \;\; 0 \le \theta_C < \frac{\pi}{2} \qquad \ldots\ldots\ldots\ldots\ldots.\ (2)$$

Now, join the points A, B, C & D to the centre O of the sphere by the dotted straight lines (As shown in the figure 1 above).

In isosceles ΔAOD,

$$OA = OD = R, \;\; \angle AOD = \phi_1$$

$$sin\frac{\phi_1}{2} = \frac{\frac{AD}{2}}{R} \quad \Rightarrow \;\; \boldsymbol{AD = 2Rsin\frac{\phi_1}{2}}$$

Similarly, in isosceles ΔCOD,

$$OC = OD = R, \;\; \angle COD = \lambda_2 - \lambda_1 \qquad \Rightarrow \; \boldsymbol{CD = 2Rsin\left(\frac{\lambda_2-\lambda_1}{2}\right)}$$

Similarly, in isosceles ΔBOC,

$$OB = OC = R, \;\; \angle BOC = \phi_2 \;\; \Rightarrow \; \boldsymbol{BC = 2Rsin\frac{\phi_2}{2}}$$

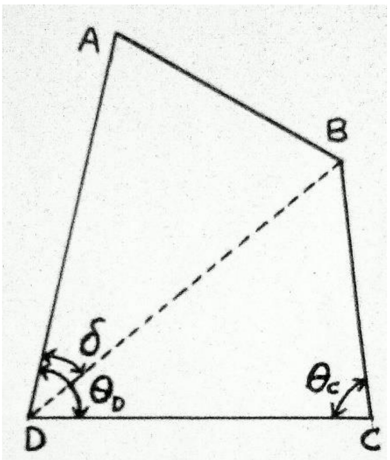

Figure 2: In plane quadrilateral ABCD, the sides AD, BC, CD & the angles θ_C & θ_D are known

Now, join the vertices B & D in the plane quadrilateral ABCD to get ΔBCD (As shown in the figure 2). Applying cosine rule in ΔBCD as follows

$$\cos\angle BCD = \frac{BC^2 + CD^2 - BD^2}{2(BC)(CD)}$$

$$\Rightarrow BD = \sqrt{BC^2 + CD^2 - 2(BC)(CD)cos\theta_C}$$

Setting all the corresponding values in above expression, we get

$$BD = \sqrt{\left(2Rsin\frac{\phi_2}{2}\right)^2 + \left(2Rsin\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)^2 - 2\left(2Rsin\frac{\phi_2}{2}\right)\left(2Rsin\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)sin\frac{\phi_2}{2}sin\left(\frac{\lambda_2-\lambda_1}{2}\right)}$$

$$= R\sqrt{4sin^2\frac{\phi_2}{2} + 4sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right) - 8sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}$$

$$= R\sqrt{2 - 2\left(1 - 2sin^2\frac{\phi_2}{2}\right)\left(1 - 2sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}$$

$$\boldsymbol{BD = R\sqrt{2 - 2cos\phi_2 cos(\lambda_2 - \lambda_1)}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(3)$$

Let $\angle ADB = \delta \;\; \Rightarrow \angle BDC = \angle ADC - \angle ADB = \theta_D - \delta, \; \angle BCD = \theta_C$. Now applying sine rule in plane ΔBCD (see figure 2 above) as follows

$$\frac{sin\angle BDC}{BC}=\frac{sin\angle BCD}{BD}\quad\Rightarrow\quad sin(\theta_D-\delta)=\frac{BC\ sin\theta_C}{BD}$$

Setting all the values (take the value of BD from (3)), we get

$$sin(\theta_D-\delta)=\frac{2Rsin\frac{\phi_2}{2}\sqrt{1-cos^2\theta_C}}{R\sqrt{2-2cos\phi_2cos(\lambda_2-\lambda_1)}}$$

$$=\frac{2sin\frac{\phi_2}{2}\sqrt{1-\left(sin\frac{\phi_2}{2}sin\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)^2}}{\sqrt{2-2cos\phi_2cos(\lambda_2-\lambda_1)}}\qquad (seeting\ value\ of\ cos\theta_C\ from\ (2)\ )$$

$$\theta_D-\delta=sin^{-1}\left(\frac{2sin\frac{\phi_2}{2}\sqrt{1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}}{\sqrt{2-2cos\phi_2cos(\lambda_2-\lambda_1)}}\right)$$

$$\delta=\theta_D-sin^{-1}\left(\frac{2sin\frac{\phi_2}{2}\sqrt{1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}}{\sqrt{2-2cos\phi_2cos(\lambda_2-\lambda_1)}}\right)$$

Since $0\le\theta_D<\frac{\pi}{2}$ hence taking cosine both the sides, we get

$$cos\delta=cos\left(\theta_D-sin^{-1}\left(\frac{2sin\frac{\phi_2}{2}\sqrt{1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}}{\sqrt{2-2cos\phi_2cos(\lambda_2-\lambda_1)}}\right)\right)$$

$$cos\delta=cos\theta_Dcos\left(sin^{-1}\left(\frac{2sin\frac{\phi_2}{2}\sqrt{1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}}{\sqrt{2-2cos\phi_2cos(\lambda_2-\lambda_1)}}\right)\right)$$
$$+sin\theta_Dsin\left(sin^{-1}\left(\frac{2sin\frac{\phi_2}{2}\sqrt{1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}}{\sqrt{2-2cos\phi_2cos(\lambda_2-\lambda_1)}}\right)\right)$$

Setting the value of $cos\theta_D$ from eq(1),

$$=sin\frac{\phi_1}{2}sin\left(\frac{\lambda_2-\lambda_1}{2}\right)cos\left(cos^{-1}\left(\sqrt{\frac{2-2cos\phi_2cos(\lambda_2-\lambda_1)-4sin^2\frac{\phi_2}{2}\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}{2-2cos\phi_2cos(\lambda_2-\lambda_1)}}\right)\right)$$
$$+\frac{2sin\frac{\phi_2}{2}\sqrt{1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}}{\sqrt{2-2cos\phi_2cos(\lambda_2-\lambda_1)}}\sqrt{1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}$$

$$= \sin\frac{\phi_1}{2}\sin\left(\frac{\lambda_2-\lambda_1}{2}\right)\sqrt{\frac{2-2\left(1-2\sin^2\frac{\phi_2}{2}\right)\left(1-2\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)-4\sin^2\frac{\phi_2}{2}+4\sin^4\frac{\phi_2}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}}$$
$$+\frac{2\sin\frac{\phi_2}{2}\sqrt{1-\sin^2\frac{\phi_2}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}}{\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}}\sqrt{1-\sin^2\frac{\phi_1}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}$$

$$= \sin\frac{\phi_1}{2}\sin\left(\frac{\lambda_2-\lambda_1}{2}\right)\frac{\sqrt{4\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)-4\sin^2\frac{\phi_2}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)+4\sin^4\frac{\phi_2}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}}{\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}}$$
$$+\frac{2\sin\frac{\phi_2}{2}\sqrt{1+\sin^4\frac{\phi_2}{2}-2\sin^2\frac{\phi_2}{2}}}{\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}}\sqrt{1-\sin^2\frac{\phi_1}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}$$

$$= \sin\frac{\phi_1}{2}\sin\left(\frac{\lambda_2-\lambda_1}{2}\right)\frac{2\sin\left(\frac{\lambda_2-\lambda_1}{2}\right)\sqrt{\left(1-\sin^2\frac{\phi_2}{2}\right)^2}}{\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}}$$
$$+\frac{2\sin\frac{\phi_2}{2}\sqrt{1-\sin^2\frac{\phi_2}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}}{\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}}\sqrt{1-\sin^2\frac{\phi_1}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}$$

$$=\frac{2\sin\frac{\phi_1}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\sqrt{\cos^4\frac{\phi_2}{2}}+2\sin\frac{\phi_2}{2}\sqrt{\left(1-\sin^2\frac{\phi_1}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-\sin^2\frac{\phi_2}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}}{\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}}$$

$$\cos\delta$$
$$=\frac{2\sin\frac{\phi_1}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\cos^2\frac{\phi_2}{2}+2\sin\frac{\phi_2}{2}\sqrt{\left(1-\sin^2\frac{\phi_1}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-\sin^2\frac{\phi_2}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}}{\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}}$$

$$\cos\delta\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}$$
$$=2\sin\frac{\phi_1}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\cos^2\frac{\phi_2}{2}$$
$$+2\sin\frac{\phi_2}{2}\sqrt{\left(1-\sin^2\frac{\phi_1}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-\sin^2\frac{\phi_2}{2}\sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}\quad \ldots\ldots\ldots\ldots.(4)$$

Now, using cosine rule in ΔADB as follows

$$\cos\angle ADB=\frac{AD^2+BD^2-AB^2}{2(AD)(BD)}$$

$$\Rightarrow\ AB^2=AD^2+BD^2-2(AD)(BD)\cos\delta$$

Setting all the corresponding values in above expression, we get

$$AB^2=\left(2R\sin\frac{\phi_1}{2}\right)^2+\left(R\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}\right)^2-2\left(2R\sin\frac{\phi_1}{2}\right)\left(R\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}\right)\cos\delta$$

$$\frac{AB^2}{R^2}=4\sin^2\frac{\phi_1}{2}+2-2\cos\phi_2\cos(\lambda_2-\lambda_1)-4\sin\frac{\phi_1}{2}\cos\delta\sqrt{2-2\cos\phi_2\cos(\lambda_2-\lambda_1)}$$

Setting the value of $cos\delta\sqrt{2-2cos\phi_2 cos(\lambda_2-\lambda_1)}$ from eq(4), we get

$$\begin{aligned}\frac{AB^2}{R^2} &= 4sin^2\frac{\phi_1}{2}+2-2cos\phi_2 cos(\lambda_2-\lambda_1)\\ &\qquad -4sin\frac{\phi_1}{2}\left(2sin\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)cos^2\frac{\phi_2}{2}\right.\\ &\qquad \left.+2sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}\right)\end{aligned}$$

$$\begin{aligned}\frac{AB^2}{R^2} &= 4sin^2\frac{\phi_1}{2}+2-2\left(1-2sin^2\frac{\phi_2}{2}\right)\left(1-2sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\\ &\qquad -4sin\frac{\phi_1}{2}\left(2sin\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\left(1-sin^2\frac{\phi_2}{2}\right)\right.\\ &\qquad \left.+2sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}\right)\end{aligned}$$

$$\begin{aligned}\frac{AB^2}{R^2} &= 4sin^2\frac{\phi_1}{2}+2-2+4sin^2\frac{\phi_2}{2}+4sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)-8sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)-8sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\\ &\quad +8sin^2\frac{\phi_1}{2}sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\\ &\quad -8sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}\end{aligned}$$

$$\begin{aligned}\frac{AB^2}{4R^2} &= sin^2\frac{\phi_1}{2}+sin^2\frac{\phi_2}{2}+sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)-2sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)-2sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\\ &\quad +2sin^2\frac{\phi_1}{2}sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\\ &\quad -2sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}\end{aligned}$$

$$\begin{aligned}&= sin^2\frac{\phi_1}{2}\left(1-2sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)+sin^2\frac{\phi_2}{2}\left(1-2sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)+sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\\ &\quad +2sin^2\frac{\phi_1}{2}sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\\ &\quad -2sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}\end{aligned}$$

$$\begin{aligned}&= sin^2\frac{\phi_1}{2}cos(\lambda_2-\lambda_1)+sin^2\frac{\phi_2}{2}cos(\lambda_2-\lambda_1)\ +sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\left(1+2sin^2\frac{\phi_1}{2}sin^2\frac{\phi_2}{2}\right)\\ &\quad -2sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}\end{aligned}$$

$$= \left(sin^2\frac{\phi_1}{2} + sin^2\frac{\phi_2}{2}\right)cos(\lambda_2-\lambda_1)\ + \left(1+2sin^2\frac{\phi_1}{2}sin^2\frac{\phi_2}{2}\right)sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)$$
$$- 2sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}$$

$$\frac{AB^2}{4R^2} = C \quad (let\ any\ arbitrary\ constant)$$

$$\Rightarrow \quad \frac{\boldsymbol{AB}}{\boldsymbol{2R}} = \sqrt{\boldsymbol{C}} \qquad \dots\dots\dots\dots\dots\dots\dots(5)$$

Where, the value of constant C is given as

$$C = \left(sin^2\frac{\phi_1}{2} + sin^2\frac{\phi_2}{2}\right)cos(\lambda_2-\lambda_1)\ + \left(1+2sin^2\frac{\phi_1}{2}sin^2\frac{\phi_2}{2}\right)sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)$$
$$- 2sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}$$

Now, in isosceles ΔAOB (see figure 1 above)

$$OA = OB = R, \ \boldsymbol{\measuredangle AOB} = \frac{\boldsymbol{length\ of\ great\ circle\ arc\ AB}}{\boldsymbol{R}}$$

$$sin\frac{\measuredangle AOB}{2} = \frac{\frac{AB}{2}}{R} = \frac{AB}{2R}$$

$$\measuredangle AOB = 2sin^{-1}\left(\frac{AB}{2R}\right)$$

Setting the value of ${AB}/{2R}$ from eq(5),

$$\measuredangle AOB = \frac{length\ of\ great\ circle\ arc\ AB}{R} = 2sin^{-1}\left(\sqrt{C}\right)$$

$$length\ of\ great\ circle\ arc\ AB = 2Rsin^{-1}\left(\sqrt{C}\right)$$

Since, great circle arc AB is the minimum distance between two given points A & B on the sphere hence

$$\boldsymbol{Minimum\ distance\ between\ the\ points\ A(\phi_1,\lambda_1)\ \&\ B(\phi_2,\lambda_2) = 2Rsin^{-1}\left(\sqrt{C}\right)}$$

Where, **C** is **HCR's constant** or **distal constant** given as

$$\boldsymbol{C = \left(sin^2\frac{\phi_1}{2} + sin^2\frac{\phi_2}{2}\right)cos(\lambda_2-\lambda_1)\ + \left(1+2sin^2\frac{\phi_1}{2}sin^2\frac{\phi_2}{2}\right)sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)}$$
$$\boldsymbol{- 2sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2\left(\frac{\lambda_2-\lambda_1}{2}\right)\right)}}$$

$$\forall \quad \boldsymbol{0 \le \phi_1\,,\phi_2, |\lambda_2-\lambda_1| \le \pi \quad \Rightarrow 0 \le C \le 1}$$

NOTE: It is obvious that the value of constant C depends on the **difference of angles of longitude** $\boldsymbol{\lambda_2-\lambda_1}$ rather than the individual values of $\boldsymbol{\lambda_1}$ & $\boldsymbol{\lambda_2}$ measured from a reference plane (like prime meridian for the globe) hence if the difference of angles of longitude is $\Delta\boldsymbol{\lambda}$ then setting $\lambda_2-\lambda_1 = \Delta\lambda$ in the expression of **distal constant C**, we get

$$\boldsymbol{C = \left(sin^2\frac{\phi_1}{2} + sin^2\frac{\phi_2}{2}\right)cos\Delta\lambda + \left(1 + 2sin^2\frac{\phi_1}{2}sin^2\frac{\phi_2}{2}\right)sin^2\frac{\Delta\lambda}{2}}$$
$$\boldsymbol{- 2sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}\sqrt{\left(1 - sin^2\frac{\phi_1}{2}sin^2\frac{\Delta\lambda}{2}\right)\left(1 - sin^2\frac{\phi_2}{2}sin^2\frac{\Delta\lambda}{2}\right)}}$$

$$\forall \quad \boldsymbol{0 \le \phi_1, \phi_2, \Delta\lambda \le \pi \quad \Rightarrow 0 \le C \le 1}$$

NOTE: It's worth noticing that the above formula of **HCR's constant C** has **symmetrical terms** i.e. if ϕ_1 & ϕ_2 are interchanged, the formula remains unchanged & hence the value of C is unchanged. It also implies that if the locations of two points for given values of latitude & longitude are interchanged, the distance between them does not change at all.

Since the equator plane divides the sphere into two equal hemispheres hence the above formula is applicable to find out the minimum distance between any two arbitrary points lying on any of two hemispheres. So for the convenience, the equator plane of the sphere should be taken in such a way that the given points lie on one of the two hemispheres resulting from division of sphere by the reference equator plane.

Since the maximum value of $sin^{-1}(\sqrt{C})$ is $\frac{\pi}{2}$ hence the max. of min. distance between two points on a sphere is

$$= 2R\left(\frac{\pi}{2}\right) = \mathbf{\pi R} = \textbf{half of the perimeter of a great circle passing through given points}$$

Case 1: If both the given points lie on the equator of the sphere then substituting $\phi_1 = \phi_2 = 0$, we get

$$C = \left(sin^2(0) + sin^2(0)\right)cos\Delta\lambda + \left(1 + 2sin^2(0)sin^2(0)\right)sin^2\frac{\Delta\lambda}{2}$$
$$- 2\sin(0)sin(0)\sqrt{\left(1 - sin^2(0)sin^2\frac{\Delta\lambda}{2}\right)\left(1 - sin^2(0)sin^2\frac{\Delta\lambda}{2}\right)}$$

$$C = sin^2\frac{\Delta\lambda}{2} \Rightarrow \sqrt{C} = \left|sin\frac{\Delta\lambda}{2}\right| = sin\frac{\Delta\lambda}{2} \qquad \left(\text{since, } 0 \le \frac{\Delta\lambda}{2} \le \frac{\pi}{2}\right)$$

Hence, the **minimum distance between the points lying on the equator of the sphere of radius R**

$$= 2Rsin^{-1}(\sqrt{C}) = 2Rsin^{-1}\left(sin\frac{\Delta\lambda}{2}\right) = 2R\frac{\Delta\lambda}{2} = \boldsymbol{R\Delta\lambda} \qquad (0 \le \Delta\lambda \le \pi)$$

The above result shows that the minimum distance between the points lying on the equator of the sphere depends only on the difference of longitudes of two given points & the radius of the sphere. This can easily be proved by using simple geometry.

If both the given points lie diametrically opposite on the equator of the sphere then substituting $\boldsymbol{\Delta\lambda = \pi}$ in above expression, the minimum distance between such points

$$R\Delta\lambda = R(\pi) = \boldsymbol{\pi R} = \text{half of the perimeter of a great circle passing through thegiven points}$$

Case 2: If both the given points lie on a great circle arc normal to the equator of the sphere then substituting $\Delta\lambda = 0$ in the formula of distal constant C, we get

$$C=\left(sin^2\frac{\phi_1}{2}+sin^2\frac{\phi_2}{2}\right)\cos(0)\ +\left(1+2sin^2\frac{\phi_1}{2}sin^2\frac{\phi_2}{2}\right)sin^2(0)$$
$$-2sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}\sqrt{\left(1-sin^2\frac{\phi_1}{2}sin^2(0)\right)\left(1-sin^2\frac{\phi_2}{2}sin^2(0)\right)}$$

$$=sin^2\frac{\phi_1}{2}+sin^2\frac{\phi_2}{2}-2sin\frac{\phi_1}{2}sin\frac{\phi_2}{2}=\left(sin\frac{\phi_1}{2}-sin\frac{\phi_2}{2}\right)^2$$

$$\Rightarrow\ \sqrt{C}=\left|sin\frac{\phi_1}{2}-sin\frac{\phi_2}{2}\right|$$

Hence, the **minimum distance between two points lying on a great circle arc normal to the equator of the sphere of radius $\boldsymbol{R}$**

$$=2Rsin^{-1}\left(\sqrt{C}\right)=\boldsymbol{2Rsin^{-1}\left|sin\frac{\phi_1}{2}-sin\frac{\phi_2}{2}\right|}\qquad(0\le\phi_1,\phi_2\le\pi)$$

Illustrative Example

Consider any two arbitrary points A & B having respective angles of latitude $\boldsymbol{\phi_1=40^o}$ & $\boldsymbol{\phi_2=75^o}$ & the difference of angles of longitude $\boldsymbol{\Delta\lambda=55^o}$ on a sphere of radius 25 cm**.** Now substituting the corresponding values, the distal constant C is given as follows

$$C=\left(sin^2\frac{40^o}{2}+sin^2\frac{75^o}{2}\right)cos55^o\ +\left(1+2sin^2\frac{40^o}{2}sin^2\frac{75^o}{2}\right)sin^2\frac{55^o}{2}$$
$$-2sin\frac{40^o}{2}sin\frac{75^o}{2}\sqrt{\left(1-sin^2\frac{40^o}{2}sin^2\frac{55^o}{2}\right)\left(1-sin^2\frac{75^o}{2}sin^2\frac{55^o}{2}\right)}$$

$$=0.279657662+0.231697606-0.394612375=0.116742892$$

Hence, the minimum distance between the given points A & B

$$=2Rsin^{-1}\left(\sqrt{C}\right)=2\times25sin^{-1}\left(\sqrt{0.116742892}\right)\approx17.43501363\ cm$$

The above result also shows that the points A & B divide the perimeter $=2\pi(25)\approx157.0796327cm$ of the great circle in two great circles arcs (one is minor arc AB of length $\approx17.43501363\ cm$ & other is major arc AB of length $\approx139.6446191\ cm$) into a ratio $\approx{}^{17.43501363}/_{139.6446191}\approx\boldsymbol{1:8}$

Conclusion: It can be concluded that this formula gives the correct values of the great circle distance because there is no approximation in the formula. This is an analytic formula to compute the minimum distance between any two arbitrary points on a sphere which is equally applicable in global positioning system. This formula is extremely useful to calculate the geographical distance between any two points on the globe for the given latitudes & longitudes. This is a highly precision formula which gives the correct values for all the distances on the tiny sphere as well as the large sphere like giant planet if the calculations are made precisely.

Note: Above articles had been derived *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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